International Journal of Engineering Sciences & Research Technology

(A Peer Reviewed Online Journal)
Impact Factor: 5.164

IJESRT

IJESRT
INTERNATIONAL JOURNAL OF ENGINEERING SCIENCES & RESEARCH TECHNOLOGY

ON THE HEPTIC EQUATION WITH FIVE UNKNOWNS $x^4+y^4-(y+x)w^3=14z^2T^5$

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DOI: 10.5281/zenodo.2540826

ABSTRACT
We obtain infinitely many non-zero integer quintuples (x, y, z, w, T) satisfying the non-homogeneous equation of degree seven with five unknowns given by $x^4+y^4-(y+x)w^3=14z^2T^5$. Various interesting properties between the solutions and special numbers are presented.

KEYWORDS: Higher degree, Heptic with five unknowns, Integer solutions.

1. INTRODUCTION
The theory of Diophantine equations offers a rich variety of fascinating problems. In particular, homogeneous and non-homogeneous equations of higher degree have aroused the interest of numerous Mathematicians since antiquity [1-3]. Particularly, in [4-10], heptic equations with three, four and five unknowns are analyzed. This paper concerns with yet another problem of determining non-trivial integral solutions of the non-homogeneous equation of seventh degree with five unknowns given by $x^4+y^4-(y+x)w^3=14z^2T^5$. A few relations between the solutions and the special numbers are presented.

2. NOTATIONS

- Polygonal number of rank n with size m

$$t_{m,n}=n\left[1+\frac{(n-1)(m-2)}{2}\right]$$

- Pyramidal number of rank n with size m

$$P_n^m=\frac{1}{6}[n(n+1)][(m-2)n+(5-m)]$$

- Centered Pyramidal number of rank n with size m

$$CP_{m,n}=\frac{m(n-1)n(n+1)+6n}{6}$$

- Stella Octangular number of rank n

$$SO_n=2n^3-n$$

- Gnomonic number of rank n

$$GNO_n=2n-1$$

- Pronic number of rank n

$$\Pr_n=n(n+1)$$

- Five dimensional Figurate number of rank n whose generating polygon is a triangle

$$F_{5,n,3}=\frac{n^5+10n^4+35n^3+50n^2+24n}{5!}$$

ISSN: 2277-9655
Impact Factor: 5.164
CODEN: IJESS7

- Four dimensional Figurate number of rank n whose generating polygon is a triangle

$$F_{4,n,3}=\frac{n^4+6n^3+11n^2+6n}{4!}$$

- Four dimensional Figurate number of rank n whose generating polygon is a square

$$F_{4,n,4}=\frac{n^4+5n^3+8n^2+4n}{12}$$

- Four dimensional Figurate number of rank n whose generating polygon is a pentagon

$$F_{4,n,5}=\frac{3n^4+10n^3+9n^2+2n}{4!}$$

- Jacobsthal number of rank n

$$J_n=\frac{1}{3}\left(2^n-(-1)^n\right)$$

- Jacobsthal-Lucas number of rank n

$$j_n=2^n+(-1)^n$$

- Kynea number of rank n

$$Ky_n=\left(2^n+1\right)^2-2$$

3. METHOD OF ANALYSIS

The Diophantine equation representing the non-homogeneous equation of degree seven is given by

$$x^4+y^4-(y+x)w^3=14z^2T^5 \tag{1}$$

Introduction of the transformations

$$x=w+z\,,\quad y=w-z \tag{2}$$

in (1) leads to

$$z^2+6w^2=7T^5 \tag{3}$$

Assume

$$T(a,b)=a^2+6b^2\quad,\ a,b>0 \tag{4}$$

Equation (3) is solved through different approaches and thus, one obtains different sets of solutions to (1).

3.1 Pattern: 1

One may write 7 as

$$7=\left(1+i\sqrt{6}\right)\left(1-i\sqrt{6}\right) \tag{5}$$

Substituting (4), (5) in (3) and employing the method of factorization, define

$$z+i\sqrt{6}w=\left(1+i\sqrt{6}\right)\left(a+i\sqrt{6}b\right)^5 \tag{6}$$

Equating real and imaginary parts in (6), we get

$$\left.\begin{aligned}z(a,b)&=a^5-30a^4b-60a^3b^2+360a^2b^3+180ab^4-216b^5\\w(a,b)&=a^5+5a^4b-60a^3b^2-60a^2b^3+180ab^4+36b^5\end{aligned}\right\} \tag{7}$$

In view of (2)

$$\left.\begin{aligned}x(a,b)&=2a^5-25a^4b-120a^3b^2+300a^2b^3+360ab^4-180b^5\\y(a,b)&=35a^4b-420a^2b^3+252b^5\end{aligned}\right\} \tag{8}$$

Thus, (4), (7), (8) represents non-zero distinct integer solutions to (1).

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Impact Factor: 5.164
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Properties:

- $x(1,b)+21600\ F_{5,b,3}-2160\ (t_{4,b})^2-6600\ CP_{6,b}-8880\ t_{4,b}-2146\ GNO_b\equiv 0(\mathrm{mod}\,3)$
- $y(a,1)-420\ F_{4,a,4}+350\ P_a^5+525\ t_{4,a}\equiv 0(\mathrm{mod}\,2)$
- $T(2^n,1)+3J_n+j_n-7=Ky_n$

3.2 Pattern: 2

One may write (3) as

$$z^2+6w^2=7T^5*1 \tag{9}$$

Write 1 as

$$1=\frac{(1+i2\sqrt{6})(1-i2\sqrt{6})}{25} \tag{10}$$

Substituting (4), (5), (10) in (9) and using the method of factorization, define

$$z+i\sqrt{6}w=\frac{1}{5}(1+i\sqrt{6})(1+i2\sqrt{6})(a+i\sqrt{6}b)^5 \tag{11}$$

Following the same procedure as in pattern:1, the corresponding non-zero distinct integral solutions to (1) are given by

$$x(A,B)=-5000A^5-90625A^4B+300000A^3B^2+1087500A^2B^3-900000AB^4-652500B^5$$
$$y(A,B)=8750A^5+21875A^4B-525000A^3B^2-262500A^2B^3+1575000AB^4+157500B^5$$
$$z(A,B)=-6875A^5-56250A^4B+412500A^3B^2+675000A^2B^3-1237500AB^4-405000B^5$$
$$w(A,B)=1875A^5-34375A^4B-112500A^3B^2+412500A^2B^3+337500AB^4-247500B^5$$
$$T(A,B)=25A^2+150B^2$$

Properties:

- $x(A,1)+10000\ P_A^5*t_{4,A}+85625\ (t_{4,A})^2-150000\ SO_A-1087500\ \mathrm{Pr}_A\equiv 0(\mathrm{mod}\,10)$
- $w(A,1)-225000\ F_{5,A,3}+1275000\ F_{4,A,3}-281250\ P_A^5-762500\ \mathrm{Pr}_A\equiv 0(\mathrm{mod}\,5)$
- $T(1,2^n)+450J_n+150j_n-175=150Ky_n$
- $T(1,B)-25$ is a Nasty number

3.3 Pattern: 3

One may write 1 as

$$1=\frac{(6-\alpha^2+i2\alpha\sqrt{6})(6-\alpha^2-i2\alpha\sqrt{6})}{(6+\alpha^2)^2} \tag{12}$$

Substituting (4), (5), (12) in (9) and employing the method of factorization, define

$$z+i\sqrt{6}w=\frac{1}{6+\alpha^2}(1+i\sqrt{6})(6-\alpha^2+i2\alpha\sqrt{6})(a+i\sqrt{6}b)^5 \tag{13}$$

Equating real and imaginary parts in (13), we get

$$\left.\begin{aligned} z(a,b,\alpha)&=\frac{1}{6+\alpha^2}[p(\alpha)\ f(a,b)-6\ q(\alpha)\ g(a,b)]\\ w(a,b,\alpha)&=\frac{1}{6+\alpha^2}[p(\alpha)\ g(a,b)+q(\alpha)\ f(a,b)]\end{aligned}\right\} \tag{14}$$

where $p(\alpha)=6-\alpha^2-12\alpha\ \ ;\ \ q(\alpha)=6-\alpha^2+2\alpha$

$f(a,b)=a^5-60a^3b^2+180ab^4\ \ ;\ \ g(a,b)=5a^4b-60a^2b^3+36b^5$

ISSN: 2277-9655
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In view of (2)

$$\left.\begin{array}{l} x(a,b,\alpha)=\frac{1}{6+\alpha^2}\left[\, f(a,b)\left(p(\alpha)+q(\alpha)\right)+g(a,b)\left(p(\alpha)-6\,q(\alpha)\right)\right] \\ y(a,b,\alpha)=\frac{1}{6+\alpha^2}\left[\, g(a,b)\left(p(\alpha)+6\,q(\alpha)\right)+f(a,b)\left(q(\alpha)-p(\alpha)\right)\right] \end{array}\right\} \quad (15)$$

Thus, (4), (14), (15) represents non-zero distinct integer solutions to (1).

To analyze the nature of solutions, one has to go for particular values of α. For simplicity, choose $\alpha=1$. Then the corresponding non-zero distinct integral solutions to (1) are given by

$$x(a,b,1)=-35a^4b+420a^2b^3-252b^5$$
$$y(a,b,1)=2a^5+25a^4b-120a^3b^2-300a^2b^3+360ab^4+180b^5$$
$$z(a,b,1)=-a^5-30a^4b+60a^3b^2+360a^2b^3-180ab^4-216b^5$$
$$w(a,b,1)=a^5-5a^4b-60a^3b^2+60a^2b^3+180ab^4-36b^5$$
$$T(a,b)=a^2+6b^2$$

Properties:

- $w(1,b,1)+z(1,b,1)+30240F_{5,b,3}-20160F_{4,b,5}-840CP_{6,b}-5040t_{4,b}\equiv 0(\mathrm{mod}\,7)$
- $x(a,1,1)+35(t_{4,a})^2-420\,\mathrm{Pr}_a\equiv 0(\mathrm{mod}\,2)$
- $y(a,a,1)+w(a,a,1)-z(a,a,1)=294\,CP_{6,b}*t_{4,b}$

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